$1/f^\alpha$ noise in the fluctuations of the spectra of tridiagonal random matrices from the β-Hermite ensemble

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The $1/f^\alpha$ noise displayed by the fluctuation of the nth unfolded eigenvalue, where n plays the role of a discrete time, was recently characterized for the classical Gaussian ensembles of $N\times N$ random matrices. It is investigated here for the β-Hermite ensemble by wavelet analysis of Monte Carlo simulated series both as a function of β and of N. When β decreases from 1 to 0, for a given and large enough N, the evolution from a $1/f$ noise at $\beta=1$ Gaussian orthogonal ensemble (GOE) to a $1/f^2$ noise at $\beta=0$ Gaussian diagonal ensemble (GDE) is heterogeneous with a $\sim 1/f^2$ noise at the finest scales and a $\sim 1/f$ noise at the coarsest ones.

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Random matrix theory (RMT) contributes significantly to quantum chaology which pertains to the statitical properties of quantum systems whose classical counterparts are chaotic [1–9]. The working definition of dynamical chaos for infinite quantum systems indeed refers to RMT [5,8]. As recalled by Prosen [8], a many-body quantum system is said to be chaotic if its excitation spectrum or some other dynamical characteristics are well described by those of ensembles of Hermitian random matrices of appropriate symmetries on certain energy or time scales. The level fluctuations of a time-reversal symmetric quantum system were, for instance, shown to coincide with those of the GOE for systems whose classical limit is chaotic [3]. The converse is, however, not necessarily always true as, for instance, the classical counterparts of quantum systems showing GOE fluctuations may be regular [10]. Quantum chaos, RMT, and statistical mechanics were brought together in particular for the special case of ideal gases [9].

The local spectral fluctuations of properly rescaled and processed eigenvalues of random matrix ensembles define universality classes in the limit of large matrix sizes which depend on the matrix symmetries and are independent on the details of the probability distributions of matrix elements. Such universality classes are for instance associated with the three fundamental Gaussian ensembles where $N\times N$ matrices are real symmetric for the GOE, Hermitian for the Gaussian unitary ensemble (GUE), and quaternion self-dual for the Gaussian symplectic ensemble (GSE). A fourth ensemble, the GDE, is made from matrices whose sole nonzero elements are diagonal with identical and independent normal distributions. Eigenvalues of Gaussian ensembles are recalled to be the equilibrium positions, at a temperature $1/\beta$, of N identical point charges on a line in 2D which interact via a logarithmic Coulomb potential and are confined by an external harmonic potential [1].

An ubiquitous characteristic of short-range correlations is the asymptotic distribution of the spacing s between consecutive energy levels of quantum systems or between successive eigenvalues of random matrices, once unfolded [1,2,5,11]. The Gaussian ensembles define, for instance, three universality classes of level repulsion at small s, $s\to 0$, $p_{W,\beta}(s)\sim s^\beta$ with $\beta=1,2,4$ for the GOE, GUE, and GSE, respectively. The latter results are not only valid for Hermitian ensembles. Ensembles of 2×2 pseudo-Hermitian random matrices, which possess real eigenvalues, with linear or quadratic level-spacing repulsions were indeed constructed by Ahmed and Jain [12]. The fluctuation properties are expected to be valid for pseudounitarily invariant $N\times N$ matrices from a line of reasoning of the Wigner surmise type [12].

A different statistic, closely related to the level density fluctuation, was considered in a series of papers [11,13–25]. The fluctuation of the nth excited state δ_n is defined as

$$\delta_n=\sum_{i=1}^{n}(s_i-1)=\varepsilon_{n+1}-\varepsilon_1-n, \tag{1}$$

where the spacing between two successive unfolded levels ε_i and ε_{i+1} is $s_i=\varepsilon_{i+1}-\varepsilon_i$. Some fluctuation characteristics of δ_n were investigated earlier by Brody *et al.* [Eq. (5.5) of Ref. [11]]. Relaño *et al.* [18] calculated correlation functions of δ_n for the GOE and the GUE. We studied similar correlations for the β-Hermite ensemble. The results are described in a related paper (in preparation). The fluctuation δ_n was recently considered as a time series of size M where n plays the role of a discrete time [14–25]. The associated power spectra were shown to display a $1/f^\alpha$ power law behavior, where $f=\frac{2\pi k}{M}$ $(k=1,\ldots,M)$ is the frequency, with an exponent α of 2 for the GDE and of 1 for the GOE, GUE, and GSE. Faleiro *et al.* [15] showed that the energy spectra of chaotic quantum systems are characterized by $1/f$ noise from random matrix theory with a power spectrum $S(k)$ varying as $N/\beta k$ for chaotic systems and as N^2/k^2 for integrable systems when the matrix size N is large and $k\ll N$. The $1/f$ behavior was shown to be robust.

Ensembles of tridiagonal random matrices with one control parameter were used to model chaotic quantum systems

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owing to their simplicity and their efficiency in numerical simulations. In that way, the level-spacing statistics of a given model can be changed from GOE-like to Poisson-like [19,26]. The β-Hermite ensemble (β-HE) [27,28], whose fluctuations characteristics were recently studied [27–30], is a family of real-symmetric tridiagonal matrices whose parameter β can be chosen at will. The multivariate eigenvalue distribution of the latter ensemble is identical with those of the corresponding Gaussian ensembles for $\beta=0,1,2,4$ respectively [27] [Eq. (4) of Ref. [30]]. All its spectral characteristics might thus be compared to those of ensembles interpolating between the classical Gaussian ensembles. The present paper investigates the $1/\mathrm{f}^{\alpha}$ noise in the spectral fluctuations of the random matrices from the β-Hermite ensemble as a function of β. Additional spectral properties, in particular at low temperature, can be found in Refs. [27–32].

A tridiagonal $N\times N$ random matrix from the β-HE is defined as

$$\mathbf{A}_{N,\beta}=\boldsymbol{\sigma}\mathbf{H}_{N,\beta}=\sigma\begin{bmatrix} H_{11} & H_{12}/\sqrt{2} & 0 & . & 0 \\ H_{12}/\sqrt{2} & H_{22} & H_{23}/\sqrt{2} & 0 & . \\ 0 & H_{23}/\sqrt{2} & . & . & 0 \\ . & 0 & . & H_{N-1,N-1} & H_{N-1,N}/\sqrt{2} \\ 0 & . & 0 & H_{N-1,N}/\sqrt{2} & H_{NN} \end{bmatrix}, \tag{2}$$

where σ is a scale factor. The $2N-1$ distinct matrix elements are independent random variables. The N diagonal elements, are independently distributed standard normal random variables with a zero mean and a standard deviation of 1. The off-diagonal element $H_{k,k+1}$ $(k=1,\ldots,N-1)$ has a chi distribution with $k\beta$ degrees of freedom whose probability density is $q_{N,\beta}(x)=2^{1-k\beta/2}x^{k\beta-1}\exp(-x^2/2)/\Gamma(k\beta/2)(x\geq 0)$.

We performed Monte Carlo calculations of random matrices from the β-Hermite ensemble both in FORTRAN and MATLAB with standard laptop computers as described in Ref. [30]. The scale parameter [Eq. (2)] was chosen so that $\langle\lambda^2\rangle=1/4$. In that way, the asymptotic eigenvalue distribution for $\beta>0$ is a Wigner semicircle of radius 1. The eigenvalue density is, to an excellent approximation, a Wigner semicircle even for moderate values of N (some tens) when β ranges between ~ 1 and ~ 5 while deviations occur both for low and for high values of β [30]. The density at high temperature (small values of β) evolves from a smooth shape intermediate between that of a Gaussian and that of a Wigner semicircle to a Wigner semicircle when the matrix size increases (Figs. 1 and 2 of Ref. [30]). At low temperature, the progressive freezing of charges around their equilibrium positions produces oscillations of the eigenvalue density around the smooth Wigner semicircle [28,30,32].

Any eigenvalue λ_k of a $N\times N$ β-Hermite matrix, which belongs to the interval $(-r,+r)$, with typical values of $r(<1)$ ranging between 0.8 and 0.9, is transformed into an unfolded eigenvalue $\lambda_k^{(u)}=\int_{-\infty}^{\lambda_k}\rho(x)dx$ from the smoothed level density $\rho(x)$. The unfolded eigenvalues are further rescaled so that the average spacing between nearest neighbors is $\langle s\rangle=1$. When the empirical cumulative distribution shows significant deviations, mostly global for $\beta<1$ [30], from a Wigner semicircle, the unfolding process was performed from a smooth eigenvalue density obtained numerically as the average of an ensemble of spectra simulated with Matlab. The simulated distributions and the "time series" investigated in the present paper were obtained altogether from simulations of 10^6 matrices with $N=25$, of 10^5 matrices with $N=200$, and of 2.10^3 matrices with $N=513$, 1000 matrices with $N=2049$, 500 matrices with $N=4097$, 500 matrices with $N=8193$, and 50 matrices with $N=32769$.

As the wavelet transform constitutes an efficient method to obtain the exponent α for the δ_n statistic [24], we chose it to follow $\alpha(\beta)$ as a function of matrix size. As signal processing methods based on wavelets are now widespread, we refer the reader to classical books [33] and we only briefly sketch the method we used [34–38]. A wide-sense stationary time series $X(t)$ can be formally written as [34,35]

$$X(t)\propto\sum_{j,k=-\infty}^{+\infty}d_X(j,k)\psi_{j,k}(t), \tag{3}$$

where the $\psi_{j,k}(t)$'s are obtained from a mother wavelet $\psi(t)$ by dyadic dilations and integer translations $\psi_{j,k}(t)=\frac{1}{2^{j/2}}\psi\left(\frac{t}{2^j}-k\right)$. The mother wavelets considered here have $M(\geq 2)$ zero moments $\int_{\mathbb{R}}t^m\psi(t)dt=0$ $(m=0,\ldots,M-1)$. The coefficients $d_X(j,k)$ of the discrete wavelet transform $d_X(j,k)=\frac{1}{2^{j/2}}\int_{-\infty}^{+\infty}X(t)\psi\left(\frac{t}{2^j}-k\right)dt$, quantify frequency details of X at scale j and at location k. When the scale j is large, the coefficient $d_X(j,k)$ captures low-frequency or coarse-scale behavior of $X(t)$. Conversely, the coefficient $d_X(j,k)$ characterizes the high- frequency or fine-scale details of $X(t)$ at small scales j [36]. If n_j is the number of coefficients at scale j, the variance $v_X(j)$ of $d_X(j,k)$, $v_X(j)$, also denoted as the mean energy of the wavelet coefficients at scale j, can be estimated from Ref. [37]:

$$v_X(j)=\left(\sum_{k=1}^{n_j}d_X(j,k)^2\right)\Bigg/ n_j \tag{4}$$

as the mean of $d_X(j,k)$ is zero by construction. The wavelet energy spectrum, defined as the set of variances $v_X(j)$, is

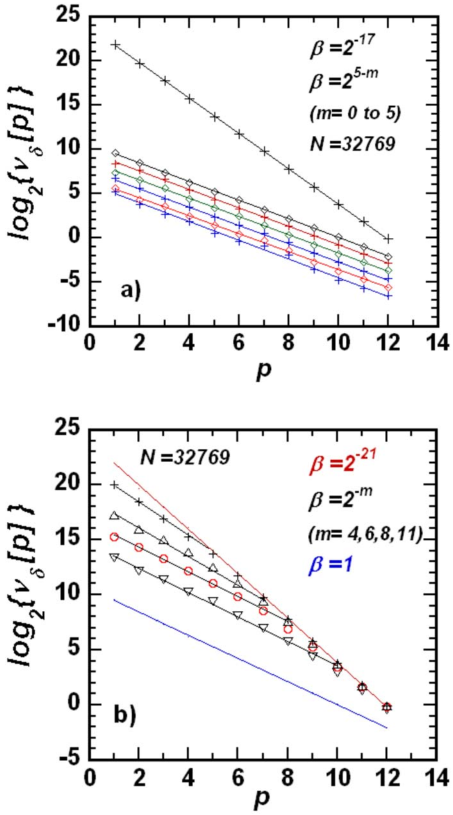

FIG. 1. (Color online) The $\log_2$ of the variance [Eq. (4)], $\log_2\{v_\delta[p]\}$, as a function of p for different values of β and of N; β decreases as indicated from bottom to top (a) from $\beta=2^5$ to $\beta=2^{-17}$ (b) from $\beta=1$ to $\beta=2^{-21}$.

related to the power spectrum of the time series $X(t)$ [34–38]. The wavelet energy spectrum summarises the spectrum information using just one value per frequency band and is of interest in particular when the power spectrum is relatively featureless in each band [35]. When the time series displays $1/f^\alpha$ noise, then in rather mild conditions [35–38]

$$v_X(j) \approx \text{const} \times 2^{j+1}\int_{2^{-(j+1)}}^{2^{-j}} \frac{df}{f^\alpha} \tag{5}$$

which gives a linear relationship between $\log_2[v_X(j)]$ and j with a slope α.

$N\times N$ β-Hermite matrices with $N=2^q+1$, with q varying from 9 to 15, were unfolded to obtain 2^{q-1} spacings. The wavelet analysis of the δ_n series was performed over $q-1$ scales with a number of coefficients decreasing from 2^{q-2} for the finest scale to 1 for the coarser using the WAVELAB software (version 850) [39]. The first set of 2^{q-2} noisy coefficients and the last coefficient were discarded and the linear regression described above was performed from the ensemble average of the second moments of the wavelet coefficients of the $q-3$ internal scales. Then, as the number of coefficients at scale j, $n_j \propto 2^{-j}$, is here $\propto 2^p$, p is equal to $-j$ except for an irrelevant shift independent of j. Linear relationships are then expected to hold between $\log_2\{v_\delta(p)\}$ and p with a slope $-\alpha$ as convincingly shown by Fig. 1. Different

TABLE I. The N dependence of the parameters β_m, defined as $\alpha(\beta_m)=1.5$ and σ_α, which are obtained by least-squares fitting the $\alpha(\beta)$ curves by Eq. (6) for a given N.

N	$\log_{10}(\beta_m)$	$10^2\beta_m$	σ_α
513	−1.23 (0.01)	5.90 (0.14)	0.87 (0.02)
2049	−1.52 (0.02)	3.02 (0.14)	0.99 (0.03)
8193	−1.79 (0.04)	1.63 (0.15)	1.01 (0.06)
32 769	−2.07 (0.04)	0.85 (0.08)	1.18 (0.06)

wavelets were used and seen to show the same bevavior of $\alpha(\beta)$ as those found with Daubechies wavelets of different indices which have compact support in the time domain and a well-localized support in the frequency domain [33]. Figure 1 shows first that the δ_n series is characterized by a $1/f$ noise for any $\beta \geq 1$ in agreement with the results found for the three classical Gaussian ensembles ($\beta=1,2,4$) [14–25]. When β decreases from 1 to 0, the noise evolves from $1/f$ at large β to $1/f^2$ when β is close to zero. An homogeneous evolution would exhibit a single intermediate $1/f^\alpha$ noise with $1<\alpha<2$ at all scales but Fig. 1(b) shows that it is heterogeneous with a $\sim 1/f^2$ noise at the finest scales and a $\sim 1/f$ noise at the coarsest ones. The analysis of the transition was nevertheless performed from the slopes of linear fits to the various curves. Therefore, a value of α intermediate between 1 and 2 is a convenient effective value but it does not necessarily mean that it results from a $1/f^\alpha$ noise at all scales. For instance, the slope obtained for $\beta=1/128$ and $N=32769$ from a linear fit of all points is $\alpha=1.50$ which is in that case the average of the slopes fitted from the zones $1 \leq p \leq 6$ and $7 \leq p \leq 12$ which are $\alpha_1=1.19$ ($\sim 1/f$) and $\alpha_2=1.83$ ($\sim 1/f^2$), respectively. Simulations and wavelet analyses were performed for $\beta=2^{-m}$ with $m=-3$ to 13. For clarity, only some of the obtained results are shown on Fig. 1(b). When β decreases in the region where α increases rapidly, the range of scales in which the $\sim 1/f^2$ noise predominates increases for a given N [Fig. 1(b)].

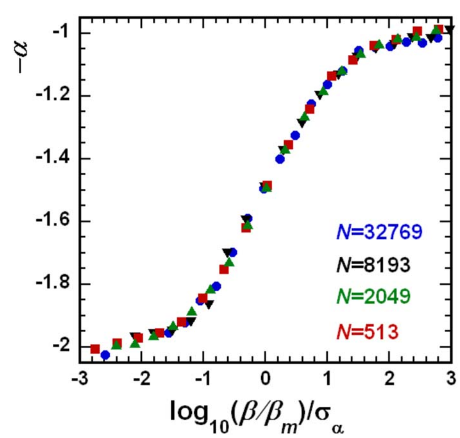

FIG. 2. (Color online) The variation of the exponent α, as a function of β and N. The four $\alpha(\beta)$ curves were rescaled with the parameters β_m and σ_α [Eq. (6)] given in Table I (full squares: $N=513$, full triangles: upwards $N=2049$, downwards $N=8193$, full circles: $N=32769$).

The analysis of the effect of the matrix size N on the $\alpha(\beta)$ curves was performed with a Daubechies wavelet of index 10. All the $\alpha(\beta)$ curves are very well described by

$$\alpha(\beta) = 1.5 - 0.5 \times \mathrm{erf}(\log_2[\beta/\beta_m]/\sigma_\alpha\sqrt{2}), \qquad (6)$$

where erf$\{x\}$ is the usual error function. The least-squares fitted parameters β_m and σ_α are given in Table I. Figure 2 shows that the four curves $\alpha\{\ln[\beta/\beta_m]/\sigma_\alpha\}$ indeed merge together and suggests that a unique growth mechanism of the $\sim 1/f^2$ fine-scales operates for any large enough matrix size. The parameter β_m decreases rather slowly with N as $\beta_m \approx 1.1/N^{0.47}$ while the apparent small increase σ_α may not be significant. In any case, the curves $\alpha(\beta)$ are shifted downwards without becoming steeper when N increases in the range investigated here. Asymptotically, the $1/f^2$ behavior occurs only at $\beta=0$ while it is the $1/f$ behavior which is the rule for $\beta>0$.

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